

A MAXIMAL FUNCTION CHARACTERIZATION FOR HARDY SPACES ASSOCIATED TO NONNEGATIVE SELF-ADJOINT OPERATORS SATISFYING GAUSSIAN ESTIMATES

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ABSTRACT. Let L be a nonnegative, self-adjoint operator satisfying Gaussian estimates on $L^2(\mathbb{R}^n)$. In this article we give an atomic decomposition for the Hardy spaces $H_{L,max}^p(\mathbb{R}^n)$ in terms of the nontangential maximal functions associated with the heat semigroup of L , and this leads eventually to characterizations of Hardy spaces associated to L , via atomic decomposition or the nontangential maximal functions. The proofs are based on a modification of technique due to A. Calderón [6].

1. INTRODUCTION

The introduction and development of Hardy spaces on Euclidean spaces \mathbb{R}^n in the 1960s played an important role in modern harmonic analysis and applications in partial differential equations. Let us recall the definition of the Hardy spaces (see [8, 14, 21, 23, 24]). Consider the Laplace operator $\Delta = -\sum_{i=1}^n \partial_{x_i}^2$ on the Euclidean spaces \mathbb{R}^n . For $0 < p < \infty$, the Hardy space $H^p(\mathbb{R}^n)$ is defined as the space of tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ for which the area integral function of f satisfies

$$(1.1) \quad Sf(x) := \left(\int_0^\infty \int_{|y-x|<t} |t^2 \Delta e^{-t^2 \Delta} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}$$

belongs to $L^p(\mathbb{R}^n)$. If this is the case, define

$$(1.2) \quad \|f\|_{H^p(\mathbb{R}^n)} := \|Sf\|_{L^p(\mathbb{R}^n)}.$$

When $p > 1$, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. For $p \leq 1$, the space $H^p(\mathbb{R}^n)$ involves many different characterizations. For example, if $f \in \mathcal{S}'(\mathbb{R}^n)$, then

$$(1.3) \quad \begin{aligned} f \in H^p(\mathbb{R}^n) &\iff \sup_{t>0} |e^{-t^2 \Delta} f(x)| \in L^p(\mathbb{R}^n) \\ &\iff \sup_{|y-x|<t} |e^{-t^2 \Delta} f(y)| \in L^p(\mathbb{R}^n) \\ &\iff f \text{ has a } (p, q) \text{ atomic decomposition } f = \sum_{j=0}^\infty \lambda_j a_j \text{ with } \sum_{j=0}^\infty |\lambda_j|^p < \infty. \end{aligned}$$

Recall that a function a supported in ball B of \mathbb{R}^n is called a (p, q) -atom, $0 < p \leq 1 \leq q \leq \infty$, $p < q$, if $\|a\|_{L^q(B)} \leq |B|^{\frac{1}{q}-\frac{1}{p}}$, and $\int_B x^\alpha a(x) dx = 0$, where α is a multi-index of order $|\alpha| \leq [n(1/p - 1)]$, the integer part of $n(1/p - 1)$ (see [8, 21, 23]).

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The theory of classical Hardy spaces has been very successful and fruitful in the past decades. However, there are important situations in which the standard theory of Hardy spaces is not applicable, including certain problems in the theory of partial differential equation which involves generalizing the Laplacian. There is a need to consider Hardy spaces that are adapted to a linear operator L , similarly to the way that the standard theory of Hardy spaces are adapted to the Laplacian. This topic has attracted a lot of attention in the last decades, and has been a very active research topic in harmonic analysis – see for example, [2, 3, 4, 10, 12, 13, 16, 17, 18, 19, 25].

In this article, we assume that L is a densely-defined operator on $L^2(\mathbb{R}^n)$ and satisfies the following properties:

(H1) L is a second order non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$;

(H2) The kernel of e^{-tL} , denoted by $p_t(x, y)$, is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies a Gaussian upper bound, that is

$$(GE) \quad |p_t(x, y)| \leq Ct^{-n/2} \exp\left(-\frac{|x-y|^2}{ct}\right)$$

for all $t > 0$, and $x, y \in \mathbb{R}^n$, where C and c are positive constants.

Given a function $f \in L^2(\mathbb{R}^n)$, consider the following area function $S_L f$ associated to the heat semigroup generated by L

$$(1.4) \quad S_L f(x) := \left(\int_0^\infty \int_{|x-y|<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Under the assumptions **(H1)** and **(H2)** of an operator L , it is known (see for example, [1, 2]) that the function S_L is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ and

$$(1.5) \quad \|S_L f\|_{L^p(\mathbb{R}^n)} \simeq \|f\|_{L^p(\mathbb{R}^n)}.$$

Definition 1.1. Suppose that an operator L satisfies **(H1)**–**(H2)**. Given $0 < p \leq 1$. The Hardy space $H_{L,S}^p(\mathbb{R}^n)$ is defined as the completion of $\{f \in L^2(\mathbb{R}^n) : \|S_L f\|_{L^p(\mathbb{R}^n)} < \infty\}$ with norm

$$\|f\|_{H_{L,S}^p(\mathbb{R}^n)} := \|S_L f\|_{L^p(\mathbb{R}^n)}.$$

To describe an atomic character of the Hardy spaces, let us recall the notion of (p, q, M) -atom associated to an operator L ([10, 16]).

Definition 1.2. Given $0 < p \leq 1 \leq q \leq \infty$, $p < q$ and $M \in \mathbb{N}$, a function $a \in L^2(\mathbb{R}^n)$ is called a (p, q, M) -atom associated to the operator L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball $B \subset \mathbb{R}^n$ such that

- (i) $a = L^M b$;
- (ii) $\text{supp } L^k b \subset B$, $k = 0, 1, \dots, M$;
- (iii) $\|(r_B^2 L)^k b\|_{L^q(\mathbb{R}^n)} \leq r_B^{2M} |B|^{1/q-1/p}$, $k = 0, 1, \dots, M$.

The atomic Hardy space $H_{L,\text{at},q,M}^p(\mathbb{R}^n)$ is defined as follows.

Definition 1.3. We will say that $f = \sum \lambda_j a_j$ is an atomic (p, q, M) -representation (of f) if $\{\lambda_j\}_{j=0}^\infty \in \ell^p$, each a_j is a (p, q, M) -atom, and the sum converges in $L^2(\mathbb{R}^n)$. Set

$$\mathbb{H}_{L,\text{at},q,M}^p(\mathbb{R}^n) := \left\{ f : f \text{ has an atomic } (p, q, M)\text{-representation} \right\},$$

with the norm $\|f\|_{\mathbb{H}_{L,\text{at},q,M}^p(\mathbb{R}^n)}$ given by

$$\inf \left\{ \left(\sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is an atomic } (p, q, M)\text{-representation} \right\}.$$

The space $H_{L,\text{at},q,M}^p(\mathbb{R}^n)$ is then defined as the completion of $\mathbb{H}_{L,\text{at},q,M}^p(\mathbb{R}^n)$ with respect to this norm.

Obviously, $H_{L,\text{at},q_2,M}^p(\mathbb{R}^n) \subseteq H_{L,\text{at},q_1,M}^p(\mathbb{R}^n)$ when $1 < q_1 \leq q_2 \leq \infty$. Under the assumption that an operator L satisfies conditions **(H1)**–**(H2)**, S. Hofmann, G. Lu, D. Mitrea, M. Mitrea and the second named author of this article obtained a $(1, 2, M)$ -atomic decomposition of the Hardy space $H_{L,S}^1(\mathbb{R}^n)$, and showed that for every number $M > 1$, the spaces $H_{L,S}^1(\mathbb{R}^n)$ and $H_{L,\text{at},2,M}^1(\mathbb{R}^n)$ coincide (see [16]). In particular,

$$\|f\|_{H_{L,S}^1(\mathbb{R}^n)} \approx \|f\|_{H_{L,\text{at},2,M}^1(\mathbb{R}^n)}.$$

A proof for $p < 1$ was shown by Duong and Li in [10], and by Jiang and Yang in [20].

Given a function $f \in L^2(\mathbb{R}^n)$, consider the non-tangential maximal function associated to the heat semigroup generated by the operator L ,

$$f_L^*(x) := \sup_{|y-x|<t} |e^{-tL} f(y)|.$$

We may define the spaces $H_{L,\max}^p(\mathbb{R}^n)$, $0 < p \leq 1$ as the completion of $\{f \in L^2(\mathbb{R}^n) : \|f_L^*\|_{L^p(\mathbb{R}^n)} < \infty\}$ with respect to L^p -norm of the non-tangential maximal function; i.e.,

$$\|f\|_{H_{L,\max}^p(\mathbb{R}^n)} := \|f_L^*\|_{L^p(\mathbb{R}^n)}.$$

It can be verified (see [16, 10]) that for every $1 < q \leq \infty$ and every number $M > \frac{n}{2}(\frac{1}{p} - 1)$, any (p, q, M) -atom a is in $H_{L,\max}^p(\mathbb{R}^n)$ and so the following continuous inclusions hold:

$$H_{L,\text{at},q,M}^p(\mathbb{R}^n) \subseteq H_{L,\max}^p(\mathbb{R}^n).$$

A natural question is to show the following continuous inclusion: $H_{L,\max}^p(\mathbb{R}^n) \subseteq H_{L,\text{at},q,M}^p(\mathbb{R}^n)$. It is known that the inclusion $H_{L,\max}^p(\mathbb{R}^n) \subseteq H_{L,\text{at},q,M}^p(\mathbb{R}^n)$ holds for some operators including Schrödinger operators with nonnegative potentials (see for example, [13, 10, 16]). However, this question is still open assuming merely that an operator L satisfies **(H1)**–**(H2)**. The aim of this article is to give an affirmative answer to this question to get an atomic decomposition directly from $H_{L,\max}^p(\mathbb{R}^n)$. We have the following result.

Theorem 1.4. *Suppose that an operator L satisfies **(H1)**–**(H2)**. Fix $0 < p \leq 1$ and $M > \frac{n}{2}(\frac{1}{p} - 1)$. Then the following three conditions are equivalent:*

(i) $f \in H_{L,\text{at},\infty,M}^p(\mathbb{R}^n)$;

(ii) Given $\alpha > 0$, $\varphi_{L,\alpha}^* f = \sup_{|y-x|<\alpha t} |\varphi(t\sqrt{L})f(y)| \in L^p(\mathbb{R}^n)$ for some even function $\varphi \in \mathcal{S}(\mathbb{R})$,

$\varphi(0) = 1$;

(iii) $G_L^*(f) = \sup_{\varphi \in \mathcal{A}} \sup_{|y-x| < t} |\varphi(t\sqrt{L})f(y)| \in L^p(\mathbb{R}^n)$, where

$$\mathcal{A} = \left\{ \varphi \in \mathcal{S}(\mathbb{R}) : \text{even functions with } \varphi(0) \neq 0, \int_{\mathbb{R}} (1+|x|)^N \sum_{k \leq N} \left| \frac{d^k}{dx^k} \varphi(x) \right|^2 dx \leq 1 \right\},$$

where N is a large number depending only on p and n .

We should mention that using the theory of tent spaces, a $(p, 2, M)$ -atomic decomposition of the Hardy space $H_{L,S}^p(\mathbb{R}^n)$ in terms of area functions was given in [10, 16]. In this article, we shall use a different argument to build a (p, ∞, M) -atomic decomposition of the Hardy spaces $H_{L,max}^p(\mathbb{R}^n)$ in terms of maximal functions. Our proof is based on a modification of technique due to A. Calderón [6], where a decomposition of the function $F(x, t) = f * \varphi_t(x)$ associated with the distribution f was given, and convolution operation of the function F played an important role in the proof. In our setting, there is, however, no analogue of convolution operation of the function $t^2 Le^{-t^2 L} f(x)$, we have to modify Calderón's construction and the geometry is conducting the analysis (see Figure 1 in Section 3). On the other hand, we do not assume that the heat kernel $p_t(x, y)$ satisfy the standard regularity condition, thus standard techniques of Calderón–Zygmund theory ([7, 23]) are not applicable. The lacking of smoothness of the kernel will be overcome in Proposition 3.1 below by using some estimates on heat kernel bounds, finite propagation speed of solutions to the wave equations and spectral theory of non-negative self-adjoint operators.

Throughout, the letter “ c ” and “ C ” will denote (possibly different) constants that are independent of the essential variables.

2. PRELIMINARIES

Recall that, if L is a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^n)$, and $E_L(\lambda)$ denotes a spectral decomposition associated with L , then for every bounded Borel function $F : [0, \infty) \rightarrow \mathbb{C}$, one defines the operator $F(L) : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ by the formula

$$(2.1) \quad F(L) := \int_0^\infty F(\lambda) dE_L(\lambda).$$

In particular, the operator $\cos(t\sqrt{L})$ is then well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from Theorem 3.4 of [9] that the integral kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$(2.2) \quad \text{supp} K_{\cos(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

By the Fourier inversion formula, whenever F is an even bounded Borel function with the Fourier transform of F , $\hat{F} \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Concretely, by recalling (2.1) we have

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^\infty \hat{F}(t) \cos(t\sqrt{L}) dt,$$

which, when combined with (2.2), gives

$$(2.3) \quad K_{F(\sqrt{L})}(x, y) = (2\pi)^{-1} \int_{|t| \geq |x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}(x, y) dt.$$

Lemma 2.1. *Let $\varphi \in C_0^\infty(\mathbb{R})$ be even, $\text{supp } \varphi \subset (-1, 1)$. Let Φ denote the Fourier transform of φ . Then for every $\kappa = 0, 1, 2, \dots$, and for every $t > 0$, the kernel $K_{(t^2 L)^\kappa \Phi(t \sqrt{L})}(x, y)$ of the operator $(t^2 L)^\kappa \Phi(t \sqrt{L})$ which was defined by the spectral theory, satisfies*

$$(2.4) \quad \text{supp } K_{(t^2 L)^\kappa \Phi(t \sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}$$

and

$$(2.5) \quad |K_{(t^2 L)^\kappa \Phi(t \sqrt{L})}(x, y)| \leq C t^{-n}$$

for all $x, y \in \mathbb{R}^n$.

Proof. For the proof, we refer it to [15] and [16]. \square

Lemma 2.2. *Assume that an operator L satisfies (H1)-(H2). Let $R > 0, s > 0$. Then for any $\epsilon > 0$, there exists a constant $C = C(s, \epsilon)$ such that*

$$\int_{\mathbb{R}^n} |K_{F(\sqrt{L})}(x, y)|^2 (1 + R|x - y|)^s dy \leq CR^n \|F(R \cdot)\|_{C^{\frac{s}{2} + \epsilon}(\mathbb{R})}^2$$

for all Borel functions F such that $\text{supp } F \subseteq [0, R]$.

Proof. For the proof, we refer the reader to Lemma 7.18, [22]. See also [11]. \square

Next we show the following result, which will be useful in the sequel.

Lemma 2.3. *Assume that an operator L satisfies (H1)-(H2). Let $\psi_i \in \mathcal{S}(\mathbb{R})$ be even functions, $\psi_i(0) = 0, i = 1, 2$. Then for every $\eta > 0$, there exists a positive constant $C = C(n, \eta, \psi_1, \psi_2)$ such that the kernel $K_{\psi_1(s \sqrt{L})\psi_2(t \sqrt{L})}(x, y)$ of $\psi_1(s \sqrt{L})\psi_2(t \sqrt{L})$ satisfies*

$$(2.6) \quad |K_{\psi_1(s \sqrt{L})\psi_2(t \sqrt{L})}(x, y)| \leq C \left(\frac{\min(s, t)}{\max(s, t)} \right) \frac{\max(s, t)^\eta}{(\max(s, t) + |x - y|)^{n+\eta}}$$

for all $t > 0$ and $x, y \in \mathbb{R}^n$.

Proof. By symmetry, it suffices to show that if $s \leq t$, then

$$(2.7) \quad |K_{\psi_1(s \sqrt{L})\psi_2(t \sqrt{L})}(x, y)| \leq C \left(\frac{s}{t} \right) \frac{t^\eta}{(t + |x - y|)^{n+\eta}}.$$

To do this, we fix $s, t > 0$ and let $\Psi(tx) = \frac{t}{s} \psi_1(sx) \psi_2(tx)$, and so $\psi_1(s \sqrt{L}) \psi_2(t \sqrt{L}) = \frac{s}{t} \Psi(t \sqrt{L})$. Let us show that

$$(2.8) \quad |K_{\Psi(t \sqrt{L})}(x, y)| \leq C t^{-n}, \quad x, y \in \mathbb{R}^n.$$

Indeed, for any $\kappa \in \mathbb{N}$, we have the relationship

$$(2.9) \quad (I + t^2 L)^{-\kappa} = \frac{1}{(\kappa - 1)!} \int_0^\infty e^{-ut^2 L} e^{-u} u^{\kappa-1} du$$

and so when $\kappa > n/4$,

$$\|(I + t^2 L)^{-\kappa}\|_{L^2(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \leq \frac{1}{(\kappa - 1)!} \int_0^\infty \|e^{-ut^2 L}\|_{L^2(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} e^{-u} u^{\kappa-1} du \leq C t^{-n/2}.$$

Now $\|(I + t^2 L)^{-\kappa}\|_{L^1(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} = \|(I + t^2 L)^{-\kappa}\|_{L^2(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \leq C t^{-n/2}$, and so

$$\|\Psi(t \sqrt{L})\|_{L^1(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)} \leq \|(I + t^2 L)^{2\kappa} \Psi(t \sqrt{L})\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \|(I + t^2 L)^{-\kappa}\|_{L^2(\mathbb{R}^n) \rightarrow L^\infty(\mathbb{R}^n)}^2.$$

Since $\psi_1 \in \mathcal{S}(\mathbb{R})$ and $\psi_1(0) = 0$, we have that $(s\lambda)^{-1} \psi_1(s\lambda) = \int_0^1 \psi'_1(s\lambda y) dy \in L^\infty(\mathbb{R})$, and then the L^2 operator norm of the last term is equal to the $L^\infty(\mathbb{R})$ norm of the function $(1 + t^2 |\lambda|)^{2m} \Psi(t \sqrt{|\lambda|}) = [(s \sqrt{|\lambda|})^{-1} \psi_1(s \sqrt{|\lambda|})][(1 + t^2 |\lambda|)^{2m} (t \sqrt{|\lambda|}) \psi_2(t \sqrt{|\lambda|})]$ which is uniformly bounded in $t > 0$. This implies that (2.8) holds.

Next, we write $F(t\lambda) = \Psi(t\lambda)(1 + t^2 \lambda^2)^m$, where $m > n/2$. Then we have $\Psi(t \sqrt{L}) = F(t \sqrt{L})(1 + t^2 L)^{-m}$. From (2.9), it can be verified that for $m > n/2$, there exist some positive constants C and c such that for every $t > 0$, the kernel $K_{(1+t^2 L)^{-m}}(x, y)$ of the operator $(1 + t^2 L)^{-m}$ satisfies

$$|K_{(1+t^2 L)^{-m}}(x, y)| \leq \frac{C}{t^n} \exp\left(-\frac{|x-y|}{c t}\right),$$

which, in combination with $(1 + \frac{|x-y|}{t}) \leq (1 + \frac{|x-z|}{t})(1 + \frac{|y-z|}{t})$, shows

$$\begin{aligned} \left| \left(1 + \frac{|x-y|}{t}\right)^{n+\eta} K_{\Psi(t \sqrt{L})}(x, y) \right| &= \left(1 + \frac{|x-y|}{t}\right)^{n+\eta} \left| \int_{\mathbb{R}^n} K_{F(t \sqrt{L})}(x, z) K_{(1+t^2 L)^{-m}}(z, y) dz \right| \\ &\leq C t^{-n} \int_{\mathbb{R}^n} |K_{F(t \sqrt{L})}(x, z)| \left(1 + \frac{|x-z|}{t}\right)^{n+\eta} dz. \end{aligned}$$

By symmetry, estimate (2.7) will be proved if we show that

$$(2.10) \quad \int_{\mathbb{R}^n} |K_{F(t \sqrt{L})}(x, z)| \left(1 + \frac{|x-z|}{t}\right)^{n+\eta} dz \leq C.$$

Let $\varphi \in C_c^\infty(0, \infty)$ be a non-negative function satisfying $\text{supp } \varphi \subseteq [\frac{1}{4}, 1]$ and let $\varphi_0 = 1 - \sum_{\ell=1}^{+\infty} \varphi(2^{-\ell} \lambda)$. So,

$$\varphi_0(\lambda) + \sum_{\ell=1}^{\infty} \varphi(2^{-\ell} \lambda) = 1, \quad \forall \lambda > 0.$$

Let $F^0(t\lambda)$ denote the function $\varphi_0(t\lambda)F(t\lambda)$ and for $\ell \geq 1$ $F^\ell(t\lambda) := \varphi(2^{-\ell} t\lambda)F(t\lambda)$. From (2.8), the proof of (2.10) reduces to estimate the following:

$$\begin{aligned} \int_{\mathbb{R}^n} |K_{F(t \sqrt{L})}(x, z)| \left(1 + \frac{|x-z|}{t}\right)^{n+\eta} dz &\leq C + \int_{\mathbb{R}^n} |K_{F^0(t \sqrt{L})}(x, z)| \left(1 + \frac{|x-z|}{t}\right)^{n+\eta} dz \\ &\quad + \sum_{\ell=1}^{\infty} \int_{|x-z| \geq t} |K_{F^\ell(t \sqrt{L})}(x, z)| \left(\frac{|x-z|}{t}\right)^{n+\eta} dz \\ (2.11) \quad &=: C + \sum_{\ell=0}^{\infty} I_\ell. \end{aligned}$$

By Lemma 2.2,

$$\begin{aligned} I_0 &\leq C t^{n/2} \left(\int_{\mathbb{R}^n} |K_{F^0(t \sqrt{L})}(x, z)|^2 \left(1 + \frac{|x-z|}{t}\right)^{3n+2\eta+1} dz \right)^{1/2} \\ &\leq C \|\delta_{1/t} F^0(t \cdot)\|_{C^{\frac{3n}{2}+2\eta+1}}. \end{aligned}$$

Since $\psi_1 \in \mathcal{S}(\mathbb{R})$ and $\psi_1(0) = 0$, we have that $(s\lambda)^{-1}\psi_1(s\lambda) = \int_0^1 \psi_1'(s\lambda y)dy \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned} I_0 &\leq C \|\varphi_0(\lambda)\Psi(\lambda)(1+\lambda^2)^m\|_{C^{\frac{3n}{2}+2\eta+1}} \\ (2.12) \quad &= \|\varphi_0(\lambda) \int_0^1 \psi_1'(s\lambda y/t)dy [\lambda\psi_2(\lambda)(1+\lambda^2)^m]\|_{C^{\frac{3n}{2}+2\eta+1}} \leq C. \end{aligned}$$

For the term I_ℓ , we use Lemma 2.2 again to obtain

$$\begin{aligned} I_\ell &\leq C t^{n/2} \left(\int_{\mathbb{R}^n} |K_{F^\ell(t\sqrt{L})}(x, z)|^2 \left(\frac{|x-z|}{t} \right)^{3n+2\eta+1} dz \right)^{1/2} \\ &\leq C t^{n/2} 2^{-\ell(3n+2\eta+1)/2} \left(\int_{\mathbb{R}^n} |K_{F^\ell(t\sqrt{L})}(x, z)|^2 \left(1 + \frac{2^\ell|x-z|}{t} \right)^{3n+2\eta+1} dz \right)^{1/2} \\ &\leq C 2^{-\ell(3n+2\eta+1)/2} 2^{\ell n/2} \|\delta_{2^\ell/t} F^\ell(t\cdot)\|_{C^{\frac{3n}{2}+2\eta+1}}. \end{aligned}$$

It can be verified that for $\psi_i \in \mathcal{S}(\mathbb{R}), i = 1, 2$,

$$\begin{aligned} \|\varphi \delta_{2^\ell} F\|_{C^{\frac{3n}{2}+2\eta+1}} &= \|\varphi(\lambda) \int_0^1 \psi_1'(2^\ell s\lambda y/t)dy [2^\ell \lambda \psi_2(2^\ell \lambda)(1+2^{2\ell}\lambda^2)^m]\|_{C^{\frac{3n}{2}+2\eta+1}} \\ &\leq C 2^{\ell(\frac{3}{2}n+2\eta+1)} 2^{-2n\ell}, \end{aligned}$$

which gives

$$\begin{aligned} \sum_{\ell=1}^{\infty} I_\ell &\leq C \sum_{\ell=1}^{\infty} C 2^{-\ell(3n+2\eta+1)/2} 2^{\ell n/2} 2^{\ell(\frac{3}{2}n+2\eta+1)} 2^{-2n\ell} \\ (2.13) \quad &\leq C \sum_{\ell=1}^{\infty} 2^{-n\ell} \leq C. \end{aligned}$$

Putting (2.12) and (2.13) into (2.11), estimate (2.10) follows readily. The proof of Lemma 2.3 is complete. \square

3. PROOF OF THEOREM 1.4

The proof of Theorem 1.4 follows the line of (ii) \Rightarrow (i) \Rightarrow (iii) \Rightarrow (ii). The proof of (i) \Rightarrow (iii) will be an adaptation of the proof of the earlier known implication of (i) \Rightarrow (ii) (see [16, 10]). Obviously, (iii) \Rightarrow (ii). The left of the proof of Theorem 1.4 is to show an implication (ii) \Rightarrow (i). To do this, we first show the following result.

Proposition 3.1. *Let $0 < p \leq 1$. Let L be a non-negative self-adjoint operator on $L^2(\mathbb{R}^n)$ satisfying Gaussian estimates (GE). Let $\varphi_i \in \mathcal{S}(\mathbb{R})$ be even functions with $\varphi_i(0) = 1$ and $\alpha_i > 0, i = 1, 2$. Then there exists a constant $C = C(n, \varphi_1, \varphi_2, \alpha_1, \alpha_2)$ such that for every $f \in L^2(\mathbb{R}^n)$, the functions $\varphi_{i,L,\alpha}^* f = \sup_{|y-x|<\alpha t} |\varphi_i(t\sqrt{L})f(y)|, i = 1, 2$, satisfy*

$$(3.1) \quad \|\varphi_{1,L,\alpha_1}^* f\|_{L^p(\mathbb{R}^n)} \leq C \|\varphi_{2,L,\alpha_2}^* f\|_{L^p(\mathbb{R}^n)}.$$

As a consequence, for any $\varphi \in \mathcal{S}(\mathbb{R})$ be even function with $\varphi(0) = 1$,

$$C^{-1} \|f_L^*\|_{L^p(\mathbb{R}^n)} \leq \|\varphi_{L,\alpha}^* f\|_{L^p(\mathbb{R}^n)} \leq C \|f_L^*\|_{L^p(\mathbb{R}^n)}, \quad \alpha > 0.$$

Proof. Recall that for any $0 < \alpha_2 \leq \alpha_1$,

$$\|\varphi_{L,\alpha_1}^* f\|_{L^p(\mathbb{R}^n)} \leq C \left(1 + \frac{\alpha_1}{\alpha_2}\right)^{n/p} \|\varphi_{L,\alpha_2}^* f\|_{L^p(\mathbb{R}^n)}$$

for any $\varphi \in \mathcal{S}(\mathbb{R})$ (Theorem 2.3, [7]). Now, we let $\psi(x) := \varphi_1(x) - \varphi_2(x)$, and then the proof of (3.1) reduces to show that

$$(3.2) \quad \|\psi_{L,1}^* f\|_{L^p(\mathbb{R}^n)} \leq C \|\varphi_{2,L,1}^* f\|_{L^p(\mathbb{R}^n)}.$$

Let us show (3.2). Let $\Psi(x) = x^{2\kappa} \Phi(x)$ where $\Phi(x)$ is the function as in Lemma 2.1 and $2\kappa > (n+1)/p$. By the spectral theory ([25]), we have

$$f = C_{\Psi, \varphi_2} \int_0^\infty \Psi(s\sqrt{L}) \varphi_2(s\sqrt{L}) f \frac{ds}{s}.$$

Therefore,

$$\psi(t\sqrt{L})f(x) = C \int_0^\infty (\psi(t\sqrt{L})\Psi(s\sqrt{L})) \varphi_2(s\sqrt{L})f(x) \frac{ds}{s}.$$

Let us denote the kernel of $\psi(t\sqrt{L})\Psi(s\sqrt{L})$ by $K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x, y)$. Then for $\lambda \in (\frac{n}{p}, 2\kappa)$,

$$(3.3) \quad \begin{aligned} & \sup_{|w|<t} |\psi(t\sqrt{L})f(x-w)| \\ &= C \sup_{|w|<t} \left| \int_{\mathbb{R}_+^{n+1}} K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z) \varphi_2(s\sqrt{L})f(z) \frac{dz ds}{s} \right| \\ &\leq C \sup_{|w|<t} \int_{\mathbb{R}_+^{n+1}} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{|x-z|}{s}\right)^{-\lambda} \frac{dz ds}{s} \\ &\leq \sup_{z,s} |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{|x-z|}{s}\right)^{-\lambda} \sup_{|w|<t} \int_{\mathbb{R}_+^{n+1}} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda \frac{dz ds}{s}. \end{aligned}$$

Next we will prove that

$$(3.4) \quad \sup_{|w|<t} \int_{\mathbb{R}_+^{n+1}} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda \frac{dz ds}{s} \leq C.$$

Once estimate (3.4) is shown, (3.2) follows. Indeed, it follows from (3.3), (3.4) and the condition $\lambda \in (\frac{n}{p}, 2\kappa)$ that

$$\begin{aligned} \|\psi_{L,1}^* f\|_{L^p(\mathbb{R}^n)} &= \left\| \sup_{|w|<t} |\psi(t\sqrt{L})f(x-w)| \right\|_{L_x^p(\mathbb{R}^n)} \leq C \left\| \sup_{z,s} |\varphi_2(s\sqrt{L})f(z)| \left(1 + \frac{|x-z|}{s}\right)^{-\lambda} \right\|_{L_x^p(\mathbb{R}^n)} \\ &\leq C \left\| \sup_{|y-x|<t} |\varphi_2(t\sqrt{L})f(y)| \right\|_{L_x^p(\mathbb{R}^n)} \\ &= C \|\varphi_{2,L,1}^* f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

where we used Theorem 2.4 of [7] in the second inequality.

Let us prove (3.4). Note that $|w| < t$. We write

$$\psi(t\sqrt{L})\Psi(s\sqrt{L}) = \begin{cases} \left(\frac{s}{t}\right)^{2\kappa} [\psi(t\sqrt{L})(t\sqrt{L})^{2\kappa}\Phi(s\sqrt{L})], & \text{if } s \leq t; \\ \left(\frac{t}{s}\right)^2 [(t\sqrt{L})^{-2}\psi(t\sqrt{L})(s\sqrt{L})^{2\kappa+2}\Phi(s\sqrt{L})], & \text{if } s > t. \end{cases}$$

We then apply Lemma 2.3 to obtain that for $\eta \in (\lambda, 2\kappa)$,

$$|K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa}, \left(\frac{t}{s}\right)^2\right) \frac{\max(s, t)^\eta}{(\max(s, t) + |x-w-z|)^{n+\eta}}.$$

This, together with the fact that

$$\int_{|u|<s} \frac{\max(s, t)^\eta}{(\max(s, t) + |u-w|)^{n+\eta}} \left(1 + \frac{|u|}{s}\right)^\lambda du \leq 2^\lambda \int_{|u|<s} \frac{\max(s, t)^\eta}{(\max(s, t) + |u-w|)^{n+\eta}} du \leq C,$$

shows

$$\begin{aligned} & \int_{\mathbb{R}^n} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda dz \\ (3.5) \quad & \leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa}, \left(\frac{t}{s}\right)^2\right) \left[1 + \int_{|u|\geq s} \frac{\max(s, t)^\eta}{(\max(s, t) + |u-w|)^{n+\eta}} \left(1 + \frac{|u|}{s}\right)^\lambda du\right]. \end{aligned}$$

To estimate the integrals over $|u| \geq s$, we note that if $s \geq t$, then we use the fact that $\eta > \lambda$ and $s + |u-w| \geq t + |u-w| \geq |w| + |u-w| \geq |u|$ to obtain

$$\begin{aligned} \int_{|u|\geq s} \frac{s^\eta}{(s + |u-w|)^{n+\eta}} \left(1 + \frac{|u|}{s}\right)^\lambda du & \leq 2^\lambda \int_{|u|\geq s} \frac{s^\eta}{(s + |u-w|)^{n+\eta}} \frac{|u|^\lambda}{s^\lambda} du \\ (3.6) \quad & \leq C \int_{|u|\geq s} \frac{s^\eta}{|u|^{n+\eta}} \frac{|u|^\lambda}{s^\lambda} du \leq C. \end{aligned}$$

If $s < t$, then from the fact that $t + |u-w| \geq |w| + |u-w| \geq |u|$ and $\eta > \lambda$,

$$\begin{aligned} \int_{|u|\geq s} \frac{t^\eta}{(t + |u-w|)^{n+\eta}} \left(1 + \frac{|u|}{s}\right)^\lambda du & \leq 2^\lambda \int_{|u|\geq s} \frac{t^\eta}{(t + |u-w|)^{n+\eta}} \frac{|u|^\lambda}{s^\lambda} du \\ (3.7) \quad & \leq C \int_{|u|\geq s} \frac{t^\eta}{|u|^{n+\eta}} \frac{|u|^\lambda}{s^\lambda} du \leq C \left(\frac{t}{s}\right)^\eta. \end{aligned}$$

Putting estimates (3.6) and (3.7) into (3.5), we have obtained that for any $|w| < t$,

$$\begin{aligned} \int_{\mathbb{R}^n} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda dz & \leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa}, \left(\frac{t}{s}\right)^2\right) \left[1 + \max\left(1, \left(\frac{t}{s}\right)^\eta\right)\right] \\ & \leq C \min\left(\left(\frac{s}{t}\right)^{2\kappa-\eta}, \left(\frac{t}{s}\right)^2\right). \end{aligned}$$

Observe that $\eta < 2\kappa$. It follows

$$\sup_{|w|<t} \int_{\mathbb{R}_+^{n+1}} |K_{\psi(t\sqrt{L})\Psi(s\sqrt{L})}(x-w, z)| \left(1 + \frac{|x-z|}{s}\right)^\lambda \frac{dz ds}{s} \leq C \int_0^\infty \min\left(\left(\frac{s}{t}\right)^{2\kappa-\eta}, \left(\frac{t}{s}\right)^2\right) \frac{ds}{s} \leq C,$$

which shows estimate (3.4), and the proof of Proposition 3.1 is end. \square

Proof of Theorem 1.4. To prove the implication (ii) \Rightarrow (i) of Theorem 1.4, from Proposition 3.1 it suffices to show that for $f \in H_{L, \max}^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, f has a (p, ∞, M) atomic representation.

We start with a suitable version of the Calderón reproducing formula. Let Φ be a function defined in Lemma 2.1, and set $\Psi(x) := x^{2M}\Phi(x)$, $x \in \mathbb{R}$. By the spectral theory ([26]), for every $f \in L^2(\mathbb{R}^n)$ one can write

$$(3.8) \quad \begin{aligned} f &= c_\Psi \int_0^\infty \Psi(t\sqrt{L})t^2Le^{-t^2L}f \frac{dt}{t} \\ &= \lim_{\epsilon \rightarrow 0} c_\Psi \int_\epsilon^{1/\epsilon} \Psi(t\sqrt{L})t^2Le^{-t^2L}f \frac{dt}{t} \end{aligned}$$

with the integral converging in $L^2(\mathbb{R}^n)$.

Set

$$\eta(x) := c_\Psi \int_1^\infty t^2x^2\Psi(tx)e^{-t^2x^2}\frac{dt}{t} = c_\Psi \int_x^\infty y\Psi(y)e^{-y^2}dy, \quad x \neq 0$$

with $\eta(0) = 1$. It follows that $\eta \in \mathcal{S}(\mathbb{R})$ is an even function, and

$$\eta(ax) - \eta(bx) = c_\Psi \int_a^b t^2x^2\Psi(tx)e^{-t^2x^2}\frac{dt}{t}.$$

By the spectral theory ([26]) again, one has

$$(3.9) \quad c_\Psi \int_a^b \Psi(t\sqrt{L})t^2Le^{-t^2L}f \frac{dt}{t} = \eta(a\sqrt{L})f(x) - \eta(b\sqrt{L})f(x).$$

Define,

$$\mathcal{M}_L f(x) := \sup_{|x-y| < 5\sqrt{nt}} (|t^2Le^{-t^2L}f(y)| + |\eta(t\sqrt{L})f(y)|).$$

By Proposition 3.1, it follows that

$$\|\mathcal{M}_L f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H_{L,\max}^p(\mathbb{R}^n)}, \quad 0 < p \leq 1.$$

Recall that \mathbb{R}_+^{n+1} denotes the upper half-space in \mathbb{R}^{n+1} . If O is an open subset of \mathbb{R}^n , then the “tent” over O , denoted by \widehat{O} , is given as $\widehat{O} := \{(x, t) \in \mathbb{R}_+^{n+1} : B(x, 4\sqrt{nt}) \subset O\}$. For $i \in \mathbb{Z}$, we define the family of sets $O_i := \{x \in \mathbb{R}^n : \mathcal{M}_L f(x) > 2^i\}$. Now let $\{Q_{ij}\}_j$ be a Whitney decomposition of O_i such that $O_i = \cup_j Q_{ij}$ and let \widehat{O}_i be a tent region. Set $\bar{e} = (1, \dots, 1) \in \mathbb{R}^n$. For every i, j , we define

$$(3.10) \quad \tilde{Q}_{ij} := \{(y, t) \in \mathbb{R}_+^{n+1} : y + 3t\bar{e} \in Q_{ij}\}.$$

It can be verified that $\widehat{O}_i \subset \cup_j \tilde{Q}_{ij}$. Indeed, for each $(y^0, t^0) \in \widehat{O}_i$, we have that $B(y^0, 4\sqrt{nt^0}) \subset O_i$. Let $\tilde{y}^0 := y^0 + 3\bar{e}t^0$. Observe that $\tilde{y}^0 \in B(y^0, 4\sqrt{nt^0})$, and then $\tilde{y}^0 \in O_i$. Then there exists some $Q_{ij_0} \subset O_i$ such that $\tilde{y}^0 \in Q_{ij_0}$, hence $(y^0, t^0) \in \tilde{Q}_{ij_0}$ and $\widehat{O}_i \subset \cup_j \tilde{Q}_{ij}$. Note that $\tilde{Q}_{ij} \cap \tilde{Q}_{ij'} = \emptyset$ when $j \neq j'$. We obtain an decomposition for \mathbb{R}_+^{n+1} as follows:

$$\mathbb{R}_+^{n+1} = \cup_i \widehat{O}_i = \cup_i \widehat{O}_i \setminus \widehat{O}_{i+1} = \cup_i \cup_j T_{ij},$$

where

$$T_{ij} := \tilde{Q}_{ij} \cap \widehat{O}_i \setminus \widehat{O}_{i+1}.$$

Using the formula (3.8), one can write

$$f = \sum_{i,j} c_\Psi \int_0^\infty \Psi(t\sqrt{L})(\chi_{T_{ij}}t^2Le^{-t^2L}f) \frac{dt}{t}$$

$$(3.11) \quad =: \sum_{i,j} \lambda_{ij} a_{ij}$$

with the sum converging in $L^2(\mathbb{R}^n)$, where $\lambda_{ij} := 2^i |Q_{ij}|^{1/p}$, $a_{ij} := L^M b_{ij}$, and

$$b_{ij} := (\lambda_{ij})^{-1} c_\Psi \int_0^\infty t^{2M} \Phi(t\sqrt{L}) (\chi_{T_{ij}} t^2 L e^{-t^2 L} f) \frac{dt}{t}.$$

Let us show that the sum (3.11) converges in $L^2(\mathbb{R}^n)$. Indeed, since for $f \in L^2(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}_+^{n+1}} |t^2 L e^{-t\sqrt{L}} f(y)|^2 \frac{dy dt}{t} \right)^{1/2} \leq C \|f\|_{L^2(\mathbb{R}^n)},$$

we use (3.11) to obtain

$$\begin{aligned} \left\| \sum_{|i|>N_1, |j|>N_2} \lambda_{ij} a_{ij} \right\|_{L^2(\mathbb{R}^n)} &= c_\Psi \left\| \sum_{|i|>N_1, |j|>N_2} \int_{\mathbb{R}_+^{n+1}} K_{(t^2 L)^M \Phi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 L e^{-t\sqrt{L}} f(y) \frac{dt}{t} \right\|_{L^2(\mathbb{R}^n)} \\ &\leq \sup_{\|g\|_2 \leq 1} \sum_{|i|>N_1, |j|>N_2} \int_{T_{ij}} |(t^2 L)^M \Phi(t\sqrt{L}) g(y) t^2 L e^{-t\sqrt{L}} f(y)| \frac{dy dt}{t} \\ &\leq C \left(\sum_{|i|>N_1, |j|>N_2} \int_{T_{ij}} |t^2 L e^{-t\sqrt{L}} f(y)|^2 \frac{dy dt}{t} \right)^{1/2} \rightarrow 0 \end{aligned}$$

as $N_1 \rightarrow \infty, N_2 \rightarrow \infty$.

Next, we will show that, up to a normalization by a multiplicative constant, the a_{ij} are (p, ∞, M) -atoms. Once the claim is established, we shall have

$$\sum_{i,j} |\lambda_{ij}|^p = \sum_{i,j} 2^{ip} |Q_{ij}| \leq C \sum_i 2^{ip} |O_i| \leq C \|f\|_{H_{L, \max}^p(\mathbb{R}^n)}^p$$

as desired.

Let us now prove that for every i, j , the function $C^{-1} a_{ij}$ is a (p, ∞, M) -atom associated with the cube $30Q_{ij}$ for some constant C . Observe that if $(y, t) \in T_{ij}$, then $B(y, 4\sqrt{nt}) \in O_i$. Denote by $\tilde{y} := y + 3t\bar{e}$, and so $\tilde{y} \in Q_{ij}$ and $B(\tilde{y}, \sqrt{nt}) \in O_i$. The fact that Q_{ij} is the Whitney cube of O_i implies that $5Q_{ij} \cap O_i^c \neq \emptyset$. Denote the side length of Q_{ij} by $\ell(Q_{ij})$. It then follows that $t \leq 3\ell(Q_{ij})$. Since $y + 3\bar{e}t \in Q_{ij}$, we have that $y \in 20Q_{ij}$. From Lemma 2.1, the integral kernel $K_{(t^2 L)^k \Phi(t\sqrt{L})}$ of the operator $(t^2 L)^k \Phi(t\sqrt{L})$ satisfies

$$\text{supp } K_{(t^2 L)^k \Phi(t\sqrt{L})} \subseteq \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.$$

This concludes that for every $k = 0, 1, \dots, M$

$$\text{supp } (L^k b_{ij}) \subseteq 30Q_{ij}.$$

It remains to show that $\|((\ell(Q_{ij})^2 L)^k b_{ij})\|_{L^\infty(\mathbb{R}^n)} \leq C(\ell(Q_{ij}))^{2M} |Q_{ij}|^{-1/p}$, $k = 0, 1, \dots, M$. When $K = 0, 1, \dots, M-1$, it reduces to show

$$(3.12) \quad \left| \int_0^\infty \int_{\mathbb{R}^n} K_{t^{2M} L^K \Phi(t^2 L)}(x, y) \chi_{T_{ij}}(y, t) t^2 L e^{-t^2 L} f(y) dy \frac{dt}{t} \right| \leq C 2^i \ell(Q_{ij})^{2(M-K)}.$$

Indeed, If $\chi_{T_{ij}}(y, t) = 1$, then $(y, t) \in (\widehat{O_{i+1}})^c$, and so $B(y, 4\sqrt{n}t) \cap (O_{i+1})^c \neq \emptyset$. Let $\bar{x} \in B(y, 4\sqrt{n}t) \cap (O_{i+1})^c$. We have that $|t^2 Le^{-t^2 L} f(y)| \leq \mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$. By Lemma 2.1,

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}^n} K_{t^{2M} L^K \Phi(t^2 L)}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \right| \\ & \leq C 2^i \left| \int_0^{c\ell(Q_{ij})} t^{2(M-K)} \int_{\mathbb{R}^n} |K_{(t^2 L)^K \Phi(t^2 L)}(x, y)| dy \frac{dt}{t} \right| \\ & \leq C 2^i \int_0^{c\ell(Q_{ij})} t^{2(M-K)} \frac{dt}{t} \\ & \leq C 2^i \ell(Q_{ij})^{2(M-K)} \end{aligned}$$

since $K = 0, 1, \dots, M-1$.

Now we consider the case $k = M$. The proof is based on a modification of technique due to A. Calderón [6]. In this case, we need to prove that for every i, j ,

$$(3.13) \quad \left| \int_0^\infty \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \right| \leq C 2^i.$$

To show (3.13), we fix x and let $d(x, Q_{ij}) < 30\sqrt{n}\ell(Q_{ij})$. We claim that the properties of the set defining $\chi_{T_{ij}}(y, t)$ imply that there exist intervals $(0, b_0), (a_1, b_1), \dots, (a_N, \infty)$, $0 < b_0 \leq a_1 < b_1 \leq \dots \leq a_N$, $1 \leq N \leq 2n+2$, such that, for $l = 0, 1, \dots, N-1$, there holds $a_{l+1} \leq 3^{2n+2}b_l$ and

- (a) $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) = 0$ for $t > a_N$;
- (b) either $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) = 0$ or $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$ for all $t \in (a_l, b_l)$;
- (c) either $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) = 0$ or $K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$ for all $t \in (0, b_0)$.

Assuming this claim for the moment, we observe that for $d(x, Q_{ij}) < 30\sqrt{n}\ell(Q_{ij})$, one can write

$$\begin{aligned} (3.14) \quad & \int_0^\infty \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \\ & = \left\{ \int_0^{b_0} + \sum_{l=1}^{N-1} \int_{a_l}^{b_l} \right\} \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \\ & + \sum_{l=1}^{N-1} \int_{b_l}^{a_{l+1}} \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \\ & = I_1(x) + I_2(x). \end{aligned}$$

To estimate $I_1(x)$, we note that if $a_l \leq a < b \leq b_l$ or $0 \leq a < b \leq b_0$, then one has either

$$\int_a^b \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} = 0,$$

or by (3.9),

$$\begin{aligned} \int_a^b \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} & = \int_a^b \Psi(t\sqrt{L}) t^2 Le^{-t^2 L} f(x) \frac{dt}{t} \\ & = \eta(a\sqrt{L})f(x) - \eta(b\sqrt{L})f(x). \end{aligned}$$

Observe that for each $a \leq t \leq b$, if $|x-y| < t$, then $\chi_{T_{ij}}(y, t) = 1$. This tells us that $(y, t) \in (\widehat{O_{i+1}})^c$, hence $B(y, 4\sqrt{nt}) \cap O_{i+1} \neq \emptyset$. Assume that $\bar{x} \in B(y, 4\sqrt{nt}) \cap (O_{i+1})^c$. From this, we have that $|x - \bar{x}| \leq |x - y| + |y - \bar{x}| < 5\sqrt{nt}$ and $\mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$. It implies that $|\eta(t\sqrt{L})f(x)| \leq \mathcal{M}_L f(\bar{x}) \leq C2^{i+1}$ for every $a \leq t \leq b$. Therefore, $|\eta(a\sqrt{L})f(x)| \leq C2^{i+1}$ and $|\eta(b\sqrt{L})f(x)| \leq C2^{i+1}$, and so $|I_1(x)| \leq C2^{i+1}$.

Consider $I_2(x)$. If $\chi_{T_{ij}}(y, t) = 1$, then $(y, t) \in (\widehat{O_{i+1}})^c$. Thus $B(y, 4\sqrt{nt}) \cap (O_{i+1})^c \neq \emptyset$. Assume that $\bar{x} \in B(y, 4\sqrt{nt}) \cap (O_{i+1})^c$. We have that $|t^2 Le^{-t^2 L} f(y)| \leq \mathcal{M}_L f(\bar{x}) \leq 2^{i+1}$. This, together with $a_{l+1} \leq cb_l$, implies that

$$\begin{aligned} \left| \int_{b_l}^{a_{l+1}} \int_{\mathbb{R}^n} K_{\Psi(t\sqrt{L})}(x, y) \chi_{T_{ij}}(y, t) t^2 Le^{-t^2 L} f(y) dy \frac{dt}{t} \right| &\leq 2^{i+1} \left| \int_{b_l}^{cb_l} \int_{\mathbb{R}^n} |K_{\Psi(t\sqrt{L})}(x, y)| dy \frac{dt}{t} \right| \\ (3.15) \qquad \qquad \qquad &\leq C2^{i+1} \int_{b_l}^{cb_l} \frac{1}{t} dt \leq C2^{i+1}, \end{aligned}$$

which yields that $|I_2(x)| \leq C2^{i+1}$.

Combining (3.14) and (3.15), we obtain (3.13). It follows that $\|a_{ij}\|_{L^\infty} \leq C|Q_{ij}|^{-1/p}$. Up to a normalization by a multiplicative constant, the a_{ij} are (p, ∞, M) -atoms.

It remains to prove the claim (a), (b) and (c). Note that $\chi_{T_{ij}}(y, t) = \chi_{\widehat{O_i}}(y, t) \cdot \chi_{(\widehat{O_{i+1}})^c}(y, t) \cdot \chi_{\tilde{Q}_{ij}}(y, t)$; Assume that $Q_{ij} = \{(y_1, \dots, y_n) : c_k \leq y_k \leq d_k, k = 1, \dots, n\}$. Then

$$\begin{aligned} \chi_{\tilde{Q}_{ij}}(y, t) &= \prod_{l=1}^n \chi_{\{c_l \leq y_l + 3t \leq d_l\}}(y, t) \\ &= \prod_{l=1}^n \chi_{\{y_l + 3t \geq c_l\}}(y, t) \cdot \chi_{\{y_l + 3t \leq d_l\}}(y, t). \end{aligned}$$

Let $\chi_l(y, t)$ be one of the characteristic functions $\chi_{\widehat{O_i}}(y, t)$, $\chi_{(\widehat{O_{i+1}})^c}(y, t)$, $\chi_{\{y_l + 3t \leq d_l\}}(y, t)$ and $\chi_{\{y_l + 3t \geq c_l\}}(y, t)$. We will prove that there exist numbers b_l and a_{l+1} , $0 < b_l \leq a_{l+1}$, $a_{l+1} \leq 3b_l$ such that

- (P) Given x , either $K_{\Psi(t\sqrt{L})}(x, y) \chi_l(y, t) = 0$ or $K_{\Psi(t\sqrt{L})}(x, y) \chi_l(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$ for all t in each of the intervals complementary to (b_l, a_{l+1}) . And for at least one of $\chi_l(y, t)$, $K_{\Psi(t\sqrt{L})}(x, y) \chi_l(y, t) = 0$ for $t > a_{l+1}$.

Then the same holds for $\chi_{T_{ij}}(y, t) = \prod_{l=1}^{2n+2} \chi_l(y, t) K_{\Psi(t\sqrt{L})}(x, y)$ in each of the intervals complementary to the union of the intervals (b_l, a_{l+1}) , which is what was asserted in the claim. Thus we merely have to prove (P). To do this, we consider four cases.

Case 1: $\chi_l(y, t) = \chi_{\{y_l + 3t \geq c_l\}}(y, t)$.

In this case, since $\text{supp } K_{\Psi(t\sqrt{L})}(x, y) \subseteq \{y : |x - y| \leq t\}$, we have that $\text{supp } K_{\Psi(t\sqrt{L})}(x, y) \subseteq \{y : x_l - t \leq y_l \leq x_l + t\}$. If $x_l \geq c_l$, then $y_l + 3t \geq x_l + 2t \geq c_l$ for any $t > 0$. This yields

$$K_{\Psi(t\sqrt{L})}(x, y) \chi_l(y, t) = K_{\Psi(t\sqrt{L})}(x, y), \quad t > 0.$$

If $x_l < c_l$, then we choose $b_l = \frac{c_l - x_l}{4}$ and $a_{l+1} = \frac{c_l - x_l}{2}$ (see Figure 1 below).

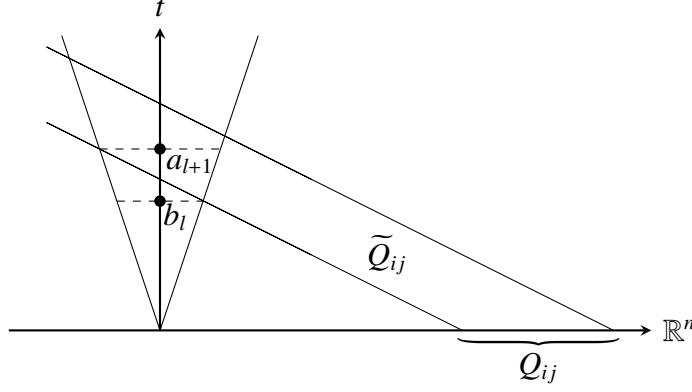


Figure 1.

In the case of $t < b_l$, we have $y_l + 3t \leq x_l + 4t < c_l$, which implies that $K_{\Psi(t\sqrt{L})}(x, y)\chi_{T_{ij}}(y, t) = 0$. In the case of $t > a_{l+1}$, we have $y_l + 3t \geq x_l + 2t > c_l$. This implies that $K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$.

Case 2: $\chi_l(y, t) = \chi_{\{y_l + 3t \leq d_l\}}(y, t)$.

Since $\text{supp } K_{\Psi(t\sqrt{L})}(x, y) \subseteq \{y : |x - y| \leq t\}$, we have that $\text{supp } K_{\Psi(t\sqrt{L})}(x, y) \subseteq \{y : x_l - t \leq y_l \leq x_l + t\}$. When $x_l \geq d_l$, we have that $y_l + 3t \geq x_l + 2t > d_l$ for any $t > 0$. This tells us

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = 0, \quad \text{for } t > 0.$$

When $x_l < d_l$, we choose $b_l = \frac{d_l - x_l}{4}$ and $a_{l+1} = \frac{d_l - x_l}{2}$. If $t < b_l$, then $y_l + 3t \leq x_l + 4t < d_l$, which implies that $K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$. If $t > a_{l+1}$, then $y_l + 3t \geq x_l + 2t > d_l$. From this, we have that $K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = 0$.

Case 3: $\chi_l(y, t) = \chi_{\widehat{O_i}}(y, t)$.

In this case, we choose $b_l = \frac{1}{5\sqrt{n}}d(x, O_i^c)$ and $a_{l+1} = \frac{1}{2\sqrt{n}}d(x, O_i^c)$. Let $|x - y| < t$. If $t < \frac{1}{5\sqrt{n}}d(x, O_i^c)$, then $d(y, O_i^c) \geq d(x, O_i^c) - |x - y| > 5\sqrt{n}t - t \geq 4\sqrt{n}t$. This tells us

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = K_{\Psi(t\sqrt{L})}(x, y)$$

for $t < \frac{1}{5\sqrt{n}}d(x, O_i^c)$. If $t > \frac{1}{2\sqrt{n}}d(x, O_i^c)$, then $d(y, O_i^c) \leq d(x, O_i^c) + d(x, y) < (2\sqrt{n} + 1)t < 4\sqrt{n}t$. Hence, if $t > \frac{1}{2\sqrt{n}}d(x, O_i^c)$, then

$$K_{\Psi(t\sqrt{L})}(x, y)\chi_l(y, t) = 0.$$

Case 4: $\chi_l(y, t) = \chi_{(\widehat{O_{i+1}})^c}(y, t)$.

In this case, we can choose $b_l = \frac{1}{5\sqrt{n}}d(x, O_{i+1}^c)$ and $a_{l+1} = \frac{1}{2\sqrt{n}}d(x, O_{i+1}^c)$. The proof can be an adaptation of the proof as in **Case 3**, and we omit the detail here.

This concludes the proof of the claim **(P)**. We have obtained the proof of an implication of (ii) \Rightarrow (i) of Theorem 1.4. The proof of Theorem 1.4 is complete. \square

In the end of this section, we consider an electromagnetic Laplacian

$$L = (i\nabla - A(x))^2 + V(x), \quad n \geq 3.$$

Recall that a measurable function V on \mathbb{R}^n is in the Kato class when

$$\limsup_{r \downarrow 0} \sup_x \int_{|x-y| < r} \frac{|V(y)|}{|x-y|^{n-2}} dy = 0,$$

while the Kato norm is defined by

$$\|V\|_K = \sup_x \int \frac{|V(y)|}{|x-y|^{n-2}} dy.$$

Proposition 3.2. *Consider an electromagnetic Laplacian*

$$L = (i\nabla - A(x))^2 + V(x), \quad n \geq 3.$$

Assume that $A \in L^2_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, and the positive and negative parts V_{\pm} of V satisfy V_+ is of Kato class, $\|V_-\|_K < c_n = \pi^{n/2}/\Gamma(n/2 - 1)$. Then for every number $M > \frac{n}{2}(\frac{1}{p} - 1)$, the spaces $H^p_{L, \max}(\mathbb{R}^n)$ and $H^p_{L, \text{at}, \infty, M}(\mathbb{R}^n)$ coincide. In particular,

$$\|f\|_{H^p_{L, \max}(\mathbb{R}^n)} \approx \|f\|_{H^p_{L, \text{at}, \infty, M}(\mathbb{R}^n)}.$$

Proof. It is known (see [5]) that under assumptions of Proposition 3.2, the operator L has a unique nonnegative self-adjoint extension, e^{-tL} is an integral operator whose kernel satisfies the Gaussian estimate (H2). Now Proposition 3.2 is a straightforward consequence of Theorem 1.4. \square

Remarks. i) Consider $L = -\Delta + V(x)$, where $V \in L^1_{\text{loc}}(\mathbb{R}^n)$ is a non-negative function on \mathbb{R}^n . It is proved in [16] that the spaces $H^1_{L, \max}(\mathbb{R}^n)$ and $H^1_{L, S}(\mathbb{R}^n)$ are equivalent, and then for every number $M \geq 1$, $H^1_{L, \max}(\mathbb{R}^n)$ and $H^1_{L, \text{at}, 2, M}(\mathbb{R}^n)$ coincide. See also [13]. However, the result of Proposition 3.2 is new.

ii) Given an operator L satisfying (H1)-(H2), we may define the spaces $H^p_{L, \text{rad}}(\mathbb{R}^n)$, $0 < p \leq 1$ as the completion of $\{f \in L^2(\mathbb{R}^n) : \|f_L^+\|_{L^p(\mathbb{R}^n)} < \infty\}$ with respect to L^p -norm of the radial maximal function; i.e.,

$$\|f\|_{H^p_{L, \text{rad}}(\mathbb{R}^n)} := \|f_L^+\|_{L^p(\mathbb{R}^n)} := \left\| \sup_{t>0} |e^{-t^2 L} f(x)| \right\|_{L^p(\mathbb{R}^n)}.$$

For every $1 < q \leq \infty$ and every number $M > \frac{n}{2}(\frac{1}{p} - 1)$, any (p, q, M) -atom a is in $H^p_{L, \text{rad}}(\mathbb{R}^n)$ and so the following continuous inclusions hold:

$$H^p_{L, \text{at}, q, M}(\mathbb{R}^n) \subseteq H^p_{L, \text{rad}}(\mathbb{R}^n).$$

See [16, 10, 13]). To the best of our knowledge, the continuous inclusion of $H^p_{L, \text{rad}}(\mathbb{R}^n) \subseteq H^p_{L, \text{at}, q, M}(\mathbb{R}^n)$ is still an open question.

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